Developing Mathematical Thinking in the 21st Century

SUMMARY

Critical and mathematical thinking are cultivated through an interactive process of discovery that uses gamification instead of rote memorization to teach higher order thinking skills in the secondary classroom. These authors explain how this approach can be used in varied contexts to increase mathematical understanding while increasing students' enthusiasm for math.

Just so we get this out of the way and the whole thing doesn't feel awkward later on, we should let you know that we're going to use the words gaming, gamers, and gamification in this article. But wait! Give us the next paragraph before moving on.

We know: You're a math teacher. You're not, for example, counting the minutes until you can play Candy Crush or Red Dead Redemption for 10 hours straight (though, alas, you might). Nor are you thinking that your students should do anything of the sort (though, alas, they might). What we will share in this article, however, are ways to use gamification to power up the teaching and learning of mathematics in the 21st century.

To be clear, when we discuss gamification, we don't mean just video games, but advancements made in the area of video games and gaming have taken learning to another level. That said, you don't need a wired classroom stocked with the latest-and-greatest technology to "gamify" anything. Gamification isn't necessarily about creating games or making learning fun either. Moreover, gamification isn't necessarily about offering rewards, points, and badges to "incentivize" students to learn.

Rather, gamification involves the strategic use of "game-based mechanics, aesthetics and game thinking to engage people, motivate action, promote learning, and solve problems" (Kapp, 2012, p. 10). We contend that the real power of gamification rests in its ability

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to inspire people, especially adolescents, to want to learn, keep learning, know what they're learning, and want to learn more. With this in mind, we offer ideas about how to harness the power of gamification and "learning like a gamer" to develop what some call mathematical thinking.

Mathematical Thinking in the 21st Century

At the heart of the Common Core State Standards in Mathematics (National Governors Association, 2010) are eight Standards for Mathematical Practice. These eight principles combine the NCTM (2000) process standards (communication, representation, reasoning and proof, connections, and problem-solving) and the National Research Council's (2001) five strands of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition). As such, the Standards for Mathematical Practice represent the aggregate of mathematical knowledge, skills, abilities, habits, and attitudes deemed essential to "producing mathematically able students well-equipped for 21st century life and career(s)" (Devlin, 2014, p. 3). Figure 1 depicts what these practices are and how they relate:

> **"Every technique and method I learned in obtaining my bachelor's and doctorate in mathematics can now be outsourced. What makes me still marketable is mathematical thinking."**
> *— Keith Devlin, Ph.D., 21st Century Mathematics Conference: Stockholm, Sweden (April 2013)*

Figure 1: Higher-Order Structure of Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
6. Attend to precision.

2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

4. Model with mathematics.
5. Use appropriate tools strategically.

7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Overarching habits of mind of a productive mathematical thinker

Reasoning and explaining

Modeling and using tools

Seeing structure and generalizing

Source: http://www.ascd.org/ascd-express/vol8/805-parker.aspx

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The real power of gamification rests in its ability to inspire people, especially adolescents, to want to learn, keep learning, know what they're learning, and want to learn more.

As a whole, these mathematical practices embody the kind of mathematical thinking important to understanding modern-day mathematics as the science of patterns:

Mathematical thinking is more than being able to do arithmetic or solve algebra problems….Mathematical thinking is a whole way of looking at things, of stripping them down to their numerical, structural, or logical essentials, and of analyzing the underlying patterns (Devlin, 2011, p. 59).

To develop the kinds of innovative mathematical thinkers needed now and in the future, Devlin recommends that we, as teachers, need to focus less on computational skills and learning procedures to solve problems, and focus more on helping students "learn how to learn" and develop "a good conceptual understanding of mathematics, its power, its scope, when and how it can be applied, and its limitations" (p. 21). So how might we do that? By *gamifying* learning and instruction.

Mathematical Thinking and Gamification

Recent developments within the field of mathematics and math education suggest that the development of mathematical thinking occurs when learning is approached as a highly interactive process of discovery and serious play rather than as a set of operations to memorize or follow (Devlin, 2012, 2011; Wallace, 2013). In a similar vein, research on the effects of video gaming in the world of work suggests that we need to seriously rethink how we're approaching teaching and learning in general — on-the-job or in classrooms. When it comes to learning in the 21st century, video gaming is clearly a game changer. Carstens and Beck (2005) argue, for example, that "games and their powerful interactivity and reinforcement of particular behaviors [and ways of thinking]" have created an entirely new generation of workers and learners who are "hardwired" in ways that significantly differ from previous generations (p. 22). They say games have not only changed how gamers think about themselves, but "how the world should work, how people should relate to one another and … the goals of life in general" (p. 23).

Currently, 91 percent of our youth in the U.S. (between the ages of 2 and 17) play video games, with 99 percent of teenage boys and 94 percent of girls playing video games in some form or another (Granic, Lobel & Engels, 2014). Given these statistics, now is definitely the time to think about this new generation of learners and how learning is accomplished. What we do know about the "gamer generation" (or those who have grown up playing videos games since the early 80s) is that when it comes to learning, they:

- require very little formal instruction
- freely trade information with other gamers
- strive to achieve meaningful goals
- face and overcome challenges that hold interest and value (Carstens & Beck, 2005; Beck & Wade, 2004)

These developments are what informed our decision to use gamification to develop mathematical thinking at the secondary level. Accordingly, in the next section of this article, we share a co-planned lesson that was taught multiple times to diverse learners in varied contexts (7th-, 11th- and 12th-grade students and college students [and nonmath majors] enrolled in a graduate-level course). Regardless of the learners' experience with, knowledge of, or interest in mathematics, all reported gaining a greater understanding and appreciation for mathematics in general and functions in particular. In this lesson, we highlight aspects of gaming used — specifically **discovery, serious play, striving toward meaningful goals** — to promote mathematical thinking around the concept of functions. In our discussion of this lesson we hope to make clear how important engagement, autonomy, mastery, and a sense of progression (through risk-free trial-and-error) are to gamification efforts of any kind.

Discovery: What is a machine?

Like all people, gamers appreciate, value, and take pride in the learning they discover themselves. Devlin (2011) suggests that learning through discovery motivates gamers "to put in the often considerable effort required to polish" their discovery but also "make good use of it" (p. 79). As such, the use of formal instruction and frontloading of information should be minimized (if not avoided). This may seem counterintuitive, but actually, it's more in line with what we know about how people learn how to problem-solve (Kapp, 2012). Using Kapp's definition, problem-solving is "any activity that involves original thinking to develop a solution, solve a dilemma, or create a product" (p. 144). One of the first things you can do to gamify your lesson is to create a dilemma or problem (or situational interest) that catches and holds your students' interest and immediately immerses students in the learning. It doesn't have to be an especially difficult or troubling situation, but it should engender sufficient situational interest. The key is to start first with mathematical concept and, as

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Devlin (2011) advises, strip it down to its numerical, structural, or logical essentials and underlying patterns. After all, mathematics is the science of patterns! (Note: Devlin says aspects of algebra, formal logic, basic set theory, elementary number theory and beginning real analysis are particularly well-suited to this task.)

For this lesson, we wrestled with how to help students discover key concepts and procedures important to the concept of functions in a fundamental and accessible, yet challenging and intriguing way. This led Derek Stoll, one of the writers of this article, to conceive of functions as machines and dynamic puzzles of sorts — something goes in, something comes out, and somewhere in between are relationships worth understanding. We must confess: Game thinking is the most important and the hardest aspect of gamification. Much like mathematical thinking, game thinking involves reducing an abstract to its bare essentials, connecting to an everyday experience that all learners would have some understanding or knowledge of, and then converting that understanding into an activity that features game-based elements such as exploration, collaboration, levels, and storytelling. We suggest doing what we did: Ask others to game-think with you. Here's the result of that thinking: To engage students and motivate action important to gamification, begin the lesson by telling a story that provides a learning goal posed as a compelling question:

> *On a day much like this one, Jay and his father are taking a walk in the park. Jay's eyes catch something in the distance. "What is THAT?" he asks.*
>
> *Jay's father replies: "Why it's a MACHINE!"*
>
> *"Huh?" Jay quizzes, "How's THAT a machine?"*

At this point, Mr. Stoll turned to the class and asked, "Hmmmm ... what IS a machine?" He prompts further, "How would you describe it? How does it work? What are some examples and non-examples of a machine?

Students record their responses on a blank sheet of paper using pictures, numbers, words, or anything else that helps them show what they understand. (Sample responses include: Does a job/ task or some kind of work, makes things easier, creates a product, has a specific purpose, a group of parts.)

As students share their responses, we do something else gamer-like: We encourage them to record anything their classmates say that helps them. There is one rule (yet another element of gamification), however: Students may not erase their answers for any reason. "Simply cross out what you no longer think," we advise. In so doing, we honor something that gamers greatly value: The ability to work cooperatively and freely trade helpful information with each other. Doing so also creates a learning environment conducive to the kind of risk-taking critical for problem-solving and innovation. All ideas (and contributions) are valued but can change, if not evolve, as more information becomes available. In this way, learners can interact with their ideas and each other without penalty or judgment. This gamified (and growth) mindset, in turn, encourages learners to continue learning and helps learners collectively and individually power up as they progress to the next level or challenge.

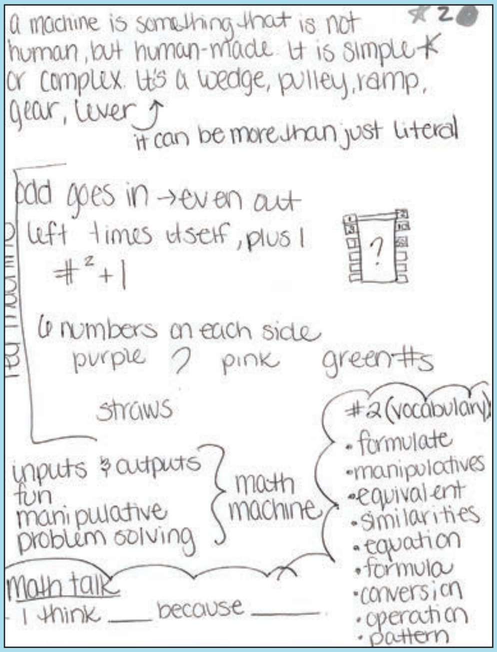
Serious play: What makes a machine a *math* machine?

To refresh, the purposes of using game-based elements and game thinking are "to engage people, motivate action, promote learning, and solve problems" (Kapp, 2012, p. 10). Gamification guru Karl Kapp clarifies, however: "Gamification is a serious approach to accelerating the curve of the learning, teaching complex subjects, and systems thinking" (p. 13). The notion of serious play — to promote worthy learning while at the same time staving off premature "death of play" — emerges as important. Ultimately, you want to purposely sequence your lesson in ways that grab and maintain your students' interest from start to finish and leave them wanting more. We suggest creating a series of progressive "tasks, missions, and activities that force the learner to synthesize knowledge from several sources" (p. 155).

At this point in the lesson, we return to Jay and his father, using story to employ another element of gamification — *assuming a role* — to invite deeper exploration of functions.

> *To help Jay understand what makes a machine a machine, Jay's father shows Jay a machine that he has been working on in the workshop. Jay is excited yet slightly confused. "This 'thing' doesn't look like a machine at all. It contains numbers, colors, different parts, and other confusing elements." Jay embarrassingly tells his father, "I am not really sure I understand what that machine is …"*

What is a Math Machine?



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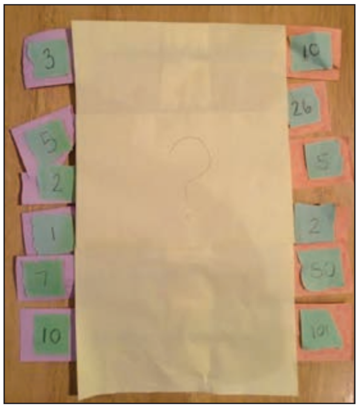

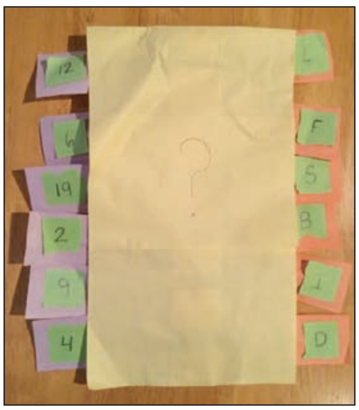

Examples of student "math machines".

"That's because it's not just any machine, it's a math machine," his father replies. "A math machine? Whoa. Math? Machine? I've never seen one before!" Jay says.

"Think back to when you were a child," Jay's father says kindly. "What did you do when you didn't understand something? What questions did you ask?"

Rather than give students a list of questions to ask, we turn to the class for their help and expertise: "If you were Jay and you didn't understand something, what questions would you ask?" Once students both identify and answer the questions raised, we return to the task at hand: "Now, let's return to this idea of a math machine: If Jay's dad says that his machine is not just any machine, but a math machine, what would make it a math machine?"

Groups of students are assigned to study math machines located throughout the classroom. Examples of those machines are provided at left.

As students examine their assigned math machine, they are prompted to think about patterns they notice. More specifically, "What types of values are going into the machines?" and "What types of values are coming out?" The idea of noticing and noting patterns is critical and fosters a modern-day definition of mathematics as the "science of patterns" (Devlin, 2011, 54).

Once students identify and analyze patterns they noticed with their respective math machines, they describe the particularities of their specific math machine and report their findings to the whole class. Words and phrases such as *input, output, uses symbols and/or data (i.e., numbers or letters*) and *shows relationships* or *it's a process* bubble up across groups. Once again, we urge students to record anything in their notes that their classmates say that helps them better understand what makes a machine not just any machine, but a *math* machine.

We then return to the story:

"Now that you have observed my math machine, do you think you can create one of your own?" Jay's father asks. Although inspired, Jay is unsure.

"Let's come up with some examples to help Jay out!" we say, but with these parameters:

- Each machine should contain at least four examples.
- All four examples should illustrate the machine's rule or function.
- The machine can use numbers or symbols.
- The machine can connect to anything of interest to them.

- As long as you can defend your work, all ideas are worthy.
- To see if your machine works with others, you will trade machines with at least two classmates. If they can figure out how your machine works, you have successfully created a math machine.

At this point in the lesson, we upped the ante in terms of using a number of features important to gamification and mathematical thinking. Initially, we used a story to invite and hold students' interest and effectively set the stage for students to become actively engaged in problem-solving. The story now provides students with a quest or challenge where multiple solutions are possible and welcomed. Students are to create their own machine, test it (by sharing it with others), get feedback, and refine. Although parameters are given, students have considerable individual choice and autonomy nonetheless.

After students have had a chance to share and test their machines, they are asked to revisit their initial understanding of a machine with the following questions in mind: (1) "What have you confirmed?" (2) "What have you revised?" and (3) "What is new that you need to add?" The development of mathematical thinking therefore occurs as the story progresses. At every step of this lesson, every student can contribute and improve or "level up" his or her performance wherever they are.

Striving toward meaningful goals

We've illuminated how to use numerous aspects of gamification to develop mathematical thinking through a highly interactive process of discovery and serious play.

No doubt, the ability to problem-solve and innovate is at a premium in today's world. Helping students learn how to work well in teams, see things in new ways, and adapt old methods to new situations, therefore, produces greater rewards for all, especially in the world of mathematics (Devlin, 2011, 21).

Ultimately, the goal of using gamification is to create learning experiences where students are invested and thus, strive to achieve meaningful goals. What's clear is that students will strive to achieve goals as long as they hold interest and value for them. So what do students say holds interest and value? The same thing that we believe makes any math teacher's heart beat: gaining an appreciation for math. Following, for example, is feedback that students provided at the end of the lesson:

"*This lesson shed a different light on math. I found value in math.*"

No doubt, the ability to problem-solve and innovate is at a premium in today's world. Helping students learn how to work well in teams, see things in new ways, and adapt old methods to new situations, therefore, produces greater rewards for all, especially in the world of mathematics.

Instead of simply learning procedures to solve problems, students develop a deep understanding of underlying concepts and justify the methods and techniques they choose to use.

"*Now when I hear the word machine, I think function and inverse.*"

"*I am not 100 percent confident when it comes to math but I will try to take more math risks!*"

"*Seems like it [math] might be worthwhile in my daily life.*"

Mathematics "*is not necessarily numbers! It's problem-solving and patterns.*"

Conclusion

We cannot predict the future with any real certainty. Still it seems reasonable to conclude that mathematical thinking will continue to prove valuable to the 21st century and beyond. It's fair to say that the demand for problem-solving, critical thinking, and innovation is nothing new. Defining mathematics as the science of patterns is, however (Devlin, 2011). With this in mind, the goal of learning (and using) mathematics in the 21st century is more about noticing, identifying and analyzing abstract patterns as they arise in the world. Instead of simply learning procedures to solve problems, students develop a deep understanding of underlying concepts and justify the methods and techniques they choose to use.

Based on what is currently known about motivation and learning, there is also something to be said for engendering a high level of student engagement not by making tasks or problems easier, but making the thinking easier. Doing so allows the struggle of all good problem-solving and critical thinking to be not only enjoyable but worth it. As the legendary basketball coach John Wooden (2005) so wisely advises, there is considerable value in making "greatness attainable by all" (p. 178). No doubt, the principles of Universal Design for Learning — namely multiple and varied means of representation, action and expression, and engagement — promote the greatness within all our students *(http://www.cast.org/udl/)*.

This changed definition spurred us to think about functions in relation to patterns of motion and thus, a machine of sorts: Something goes in, something comes out, and somewhere in between are patterns (i.e., rules, functions, and hypotheses) worth discovering and testing. To this end, we contend this modern-day view of mathematics calls for both a changed "end game" and game plan. Mathematical thinking isn't taught. Rather, it's gained through learning experiences that feature some of what video games do especially well: 1) sufficiently catching and holding students' interest; 2) keeping overt telling and/or formal instruction to a minimum; 3) encouraging learning with and from other students; 4) communicating that everyone can play regardless of their current level of knowledge and skill, that everyone has something to contribute, that risk is necessary, and that failure doesn't hurt; and 5) providing multiple and varied opportunities for every learner to improve, advance, and/or level up in meaningful ways.

If students are also hardwired to learn differently — as the research on video gaming and gamers currently suggests — we have good reason to rethink how we approach the learners now sitting in our classrooms, K-12. They've changed, but have we? No doubt, the strategic use of game-based learning is more likely to inspire these learners to want to learn, keep learning, know what they're learning, and want to learn more.

Certainly, we can choose to ignore or deny the call for change. But if we do, longstanding problems of student motivation and boredom common in middle and high school classrooms are likely to create even bigger challenges as we ask more of our students (Mitchell, 1993). For most adolescents (and people in general), the development of mathematical thinking is not easy or natural (Genovese, 2003). In fact, this is one of many reasons why we need formal education and teachers like you. We believe the strategic use of gamification provides us an especially powerful antidote. Given what is gained and by whom, using gamification to power up the teaching and learning of math in your classroom is an investment worth making.

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